Dynamic Dictionary of Mathematical Functions (DDMF)

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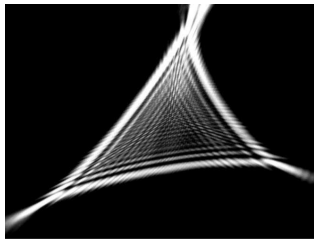
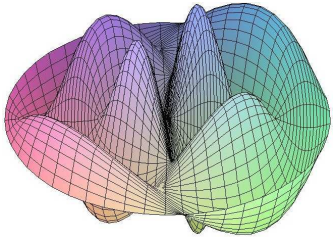
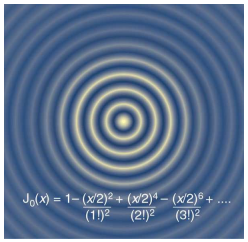
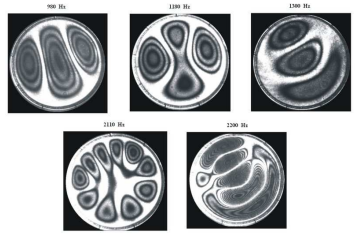
Special Functions: From Physics to Applied Mathematics

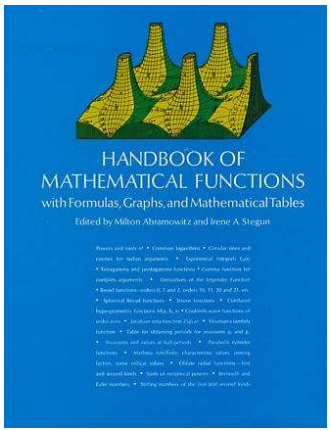
Airy function

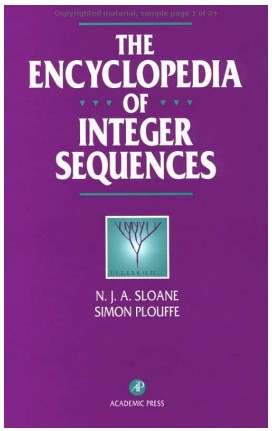



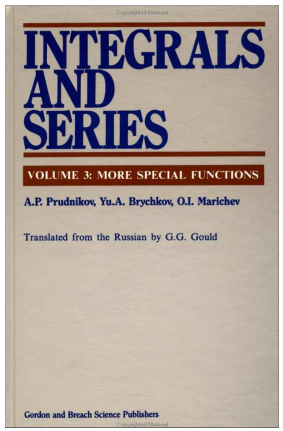


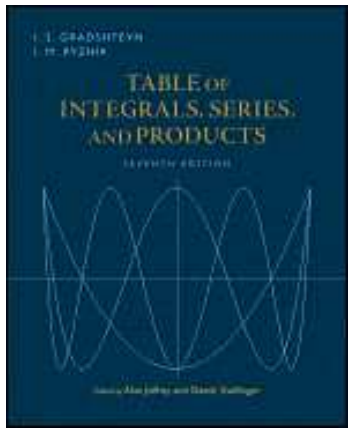
Bessel function

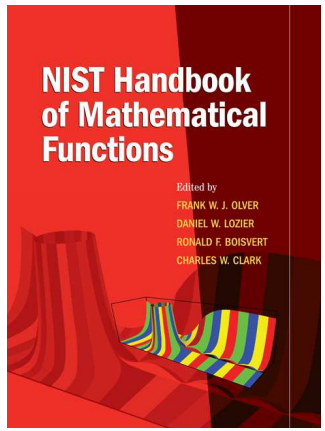
Own theory developed in mathematical analysis (19th and 20th centuries).











Algorithms developed in *Computer Algebra* since the 1980's.

Looking Inside PBM's "Integrals and Series, Vol. 2"

6. $$\int_0^\infty x^{-1/2}\cos b\sqrt{x}\,J_0(cx)\,dx =$$

$$= \frac{\pi b}{8c}\left\{Y_{1/4}(z)\left[Y_{1/4}(z)-J_{1/4}(z)\right]-J_{1/4}(z)\left[J_{1/4}(z)+Y_{1/4}(z)\right]\right\}$$

$[z = b^2/(8c);\ b,\ c>0]$

7. $$\int_0^\infty x^{-1/2}\sin b\sqrt{x}\begin{Bmatrix}\sin cx\\ \cos cx\end{Bmatrix}J_0(cx)\,dx = \sqrt{\frac{\pi}{2c}}\cos\left(\frac{b^2}{16c}\pm\frac{\pi}{4}\right)J_0\left(\frac{b^2}{16c}\right)$$

$[b,\ c>0]$.

8. $$\int_0^\infty x^{-1/2}\cos\sqrt{x}\begin{Bmatrix}\sin cx\\ \cos cx\end{Bmatrix}J_0(cx)\,dx =$$

$$= -\frac{1}{2}\sqrt{\frac{\pi}{2c}}\left[\begin{Bmatrix}\cos t\\ \sin t\end{Bmatrix}J_0\left(\frac{b^2}{16c}\right)\mp\begin{Bmatrix}\sin t\\ \cos t\end{Bmatrix}Y_0\left(\frac{b^2}{16c}\right)\right]$$

$[t = b^2/(16c)-\pi/4;\ b,\ c>0]$.

2.12.20. Integrals containing $x^\alpha\begin{Bmatrix}\sin(bx+a/x)\\ \cos(bx+a/x)\end{Bmatrix}J_\nu(cx)$.

1. $$\int_0^\infty x^{\alpha-1}\begin{Bmatrix}\sin(b/x)\\ \cos(b/x)\end{Bmatrix}J_\nu(cx)\,dx = 2^{\alpha-\delta-1}b^\delta c^{\delta-\alpha}\Gamma\begin{bmatrix}(\nu+\alpha-\delta)/2\\ 1+(\delta+\nu-\alpha)/2\end{bmatrix}\times$$

$$\times {}_0F_3\left(1-\frac{\nu+\alpha-\delta}{2},\ \frac{1}{2}+\delta,\ 1+\frac{\delta+\nu-\alpha}{2};\ \frac{b^2c^2}{16}\right)\mp$$

$$\mp b^{\alpha+\nu}\left(\frac{c}{2}\right)^\nu\begin{Bmatrix}\sin(\alpha+\nu)\pi/2\\ \cos(\alpha+\nu)\pi/2\end{Bmatrix}\begin{bmatrix}-\nu-\alpha\\ 1+\nu\end{bmatrix}{}_0F_3\left(1+\frac{\alpha+\nu}{2},\ 1+\nu,\ \frac{1+\alpha+\nu}{2};\ \frac{b^2c^2}{16}\right)$$

$\left[b,\ c>0;\ -1-\mathrm{Re}\,\nu<\mathrm{Re}\,\alpha<\delta+3/2;\ \delta=\begin{Bmatrix}1\\ 0\end{Bmatrix}\right]$.

2. $$\int_0^\infty \frac{1}{x}\sin bx\begin{Bmatrix}\sin(a/x)\\ \cos(a/x)\end{Bmatrix}J_\nu(cx)\,dx =$$

$$= \frac{\pi}{2}J_\nu(z_+\sqrt{a})\left[\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix}J_\nu(z_-\sqrt{a})\pm\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix}Y_\nu(z_-\sqrt{a})\right]+$$

$$+\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix}I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a}) \qquad \left[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}\right].$$

3. $$\int_0^\infty \frac{1}{x}\cos bx\begin{Bmatrix}\sin(a/x)\\ \cos(a/x)\end{Bmatrix}J_\nu(cx)\,dx =$$

$$= \pm\frac{\pi}{2}J_\nu(z_+\sqrt{a})\left[\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix}J_\nu(z_-\sqrt{a})\mp\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix}Y_\nu(z_-\sqrt{a})\right]\mp$$

$$\mp\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix}I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a}) \qquad \left[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}\right].$$

4. $$\int_0^\infty \frac{1}{x}\begin{Bmatrix}\sin(bx-a/x)\\ \cos(bx-a/x)\end{Bmatrix}J_\nu(cx)\,dx = 2\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix}I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a})$$

$\left[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}\right]$.

5. $$\int_0^\infty \frac{1}{x}\begin{Bmatrix}\sin(bx+a/x)\\ \cos(bx+a/x)\end{Bmatrix}J_\nu(cx)\,dx =$$

$$= J_\nu(z_+\sqrt{a})\left[\pm\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix}J_\nu(z_-\sqrt{a})-\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix}Y_\nu(z_-\sqrt{a})\right]$$

$\left[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}\right]$.

2.12.21. Integrals of $A(x)\begin{Bmatrix}\sin(b\sqrt{a^2-x^2})\\ \cos(b\sqrt{a^2-x^2})\end{Bmatrix}J_\nu(cx)$.

1. $$\int_0^a x^{\nu+1}\sin(b\sqrt{a^2-x^2})\,J_\nu(cx)\,dx =$$

$$= \sqrt{\frac{\pi}{2}}a^{\nu+3/2}bc^\nu(b^2+c^2)^{-(2\nu+3)/4}J_{\nu+3/2}(a\sqrt{b^2+c^2})$$

$[a>0;\ \mathrm{Re}\,\nu>-1]$.

2. $$\int_0^a x\sin(b\sqrt{a^2-x^2})\,J_0(cx)\,dx = \frac{ab}{b^2+c^2}\left[\frac{\sin(a\sqrt{b^2+c^2})}{a\sqrt{b^2+c^2}}-\cos(a\sqrt{b^2+c^2})\right]$$

$[a>0]$.

3. $$\int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}J_\nu(cx)\,dx = \frac{\pi}{2}J_{\nu/2}\left(\frac{a\sqrt{b^2+c^2}+ab}{2}\right)J_{\nu/2}\left(\frac{a\sqrt{b^2+c^2}-ab}{2}\right)$$

$[a>0;\ \mathrm{Re}\,\nu>-1]$.

4. $$\int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}J_1(cx)\,dx = \frac{1}{ac}\cos ab-\frac{1}{ac}\cos(a\sqrt{b^2+c^2})$$

$[a>0]$.

5. $$\int_0^a x^{\nu+1}\frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}J_\nu(cx)\,dx =$$

$$= \sqrt{\frac{\pi}{2}}a^{\nu+1/2}c^\nu(b^2+c^2)^{-(2\nu+1)/4}J_{\nu+1/2}(a\sqrt{b^2+c^2}) \qquad [a>0;\ \mathrm{Re}\,\nu>-1].$$

6. $$\int_0^a x\frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}J_0(cx)\,dx = \frac{\sin(a\sqrt{b^2+c^2})}{\sqrt{b^2+c^2}} \qquad [a>0].$$

2.12.22. Integrals of $A(x)\begin{Bmatrix}\sin(b\sqrt{x^2-a^2})\\ \cos(b\sqrt{x^2-a^2})\end{Bmatrix}J_\nu(cx)$.

1. $$\int_a^\infty \sin(b\sqrt{x^2-a^2})\,J_1(cx)\,dx = \frac{b}{c}(c^2-b^2)_+^{-1/2}\cos(a\sqrt{c^2-b^2})$$

$[a,\ b,\ c>0;\ b\neq c]$.

2. $$\int_a^\infty x^{1-\nu}\sin(b\sqrt{x^2-a^2})\,J_\nu(cx)\,dx =$$

$$= \sqrt{\frac{\pi}{2}}a^{3/2-\nu}bc^{-\nu}(c^2-b^2)_+^{(2\nu-3)/4}J_{\nu-3/2}(a\sqrt{c^2-b^2})$$

$[a>0;\ b\neq c;\ \mathrm{Re}\,\nu>-1/2]$.

Dynamic Dictionary of Mathematical Functions

DDMF = Mathematical Handbooks + Computer Algebra + Web

- Public dictionary online:

→ `http://ddmf.msr-inria.inria.fr/`
→ *DDMF* is not just a DB/collection of texts!

- New computer-algebra algorithms to generate more formulas on (Linear) Special Functions:

→ linear differential equation as a data structure,
→ implementation: *Algolib*.

- Tools for Dynamic Mathematics on the Web:

→ interactivity + incremental computations,
→ implementation: *DynaMoW*.

Ongoing & Future Work

- Certified uniform numerics [Benoit, Joldeş, Mezzarobba].
- Branch cuts [Chyzak, Davenport, Koutschan, Salvy].
- Parametrised functions [Chyzak, Koutschan]: Bessel, Kummer, ...
- More integral transforms [Labahn, Peasgood, Salvy].
- Plots: definition domains, automatic ranges, navigation.
- More explicit asymptotics.
- Special sequences and orthogonal polynomials.
- Sums and integrals: fast algos, multiple $\sum$s and $\int$s.